Representing Cyclic Structures as Nested Datatypes

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Joint work with

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Motivation

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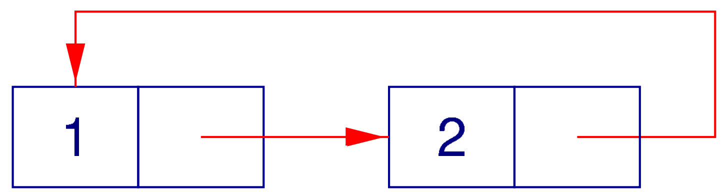


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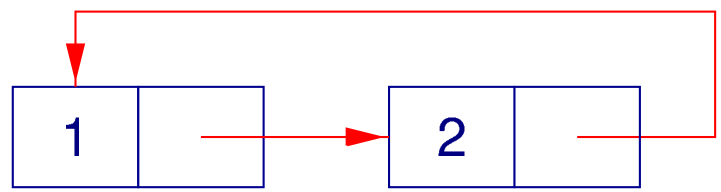


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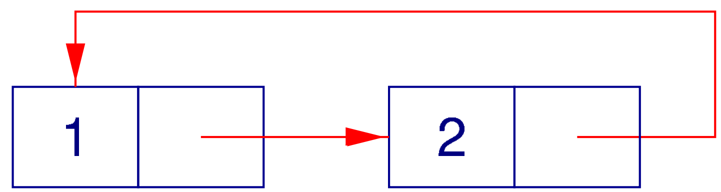


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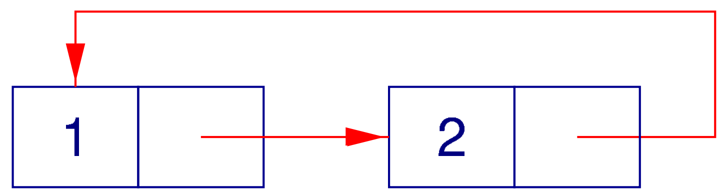


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▷ Allows to represent infinite structures in finite memory

▷ Problem: No support for manipulating cyclic structures

Problems on the Usual Approach

- ▷ No support for manipulating cyclic structures
- ▷ E.g. ··· destructing the cyclic structure!

```
map (+1) cycle  ==>  [2,3,2,3,2,3,2,3,....
```

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- ▷ **Q.** Can we represent cyclic structures inductively? i.e. by algebraic datatypes
- ▷ Merit: explicitly manipulate cyclic structures

  either directly or using generic operations like `fold`

Fegaras-Sheard Approach

▷ Cyclic lists as Mixed-variant Datatype by Fegaras, Sheard (POPL'96):

```
data CList = Nil
           | Cons Int CList
           | Rec (CList -> CList)
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▷ Examples:

```
clist1 = Rec (\ xs -> Cons 1 (Cons 2 xs))
clist2 = Cons 1 (Rec (\ xs -> Cons 2 (Cons 3 xs)))
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- Examples:

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```

- Functions manipulating these representations must unfold `Rec`-structures.

```
cmap :: (Int -> Int) -> CList -> CList
cmap g Nil         = Nil
cmap g (Cons x xs) = Cons (g x) (cmap g xs)
cmap g (Rec f)     = cmap g (f (Rec f))
```

- Implicit axiom: `Rec f = f (Rec f)`

Fegaras-Sheard Approach: Problem

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- Functions manipulating cyclic lists must unwind them

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- ▷ There is a "blackhole"

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empty = Rec (\ xs -> xs)
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- ▷ The representation is not unique:

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clist1' = Rec (\ xs -> Rec (\ ys ->
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```

- ▷ The semantic category has to be algebraically compact (e.g. **CPO**) for mixed-variant types to make semantic sense.

$$\boldsymbol{L} \;\cong\; \mathbf{1} + \mathbb{Z} \times \boldsymbol{L} + (\boldsymbol{L} \rightarrow \boldsymbol{L})$$

Our Analysis

```
data CList = Nil
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           | Rec (CList -> CList)
```

- ▷ The same problem has already appeared in "Higher-Order Abstract Syntax" (HOAS)
- ▷ Induction on function space?

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- ▷ Induction on function space?
- ▷ The same solution was proposed in FP and in semantics

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  Bird and Paterson: *De Bruijn Notation as a Nested Datatype*, JFP'99

  Fiore,Plotkin and Turi: *Abstract Syntax and Variable Binding*, LICS'99
- ▷ Represent lambda terms by a nested datatype
- ▷ Use a kind of de Bruijn notation

Cyclic Lists as Nested Datatype

▷ Our Proposal:

```
data CList a = Var a
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             | RCons Int (CList (Maybe a))
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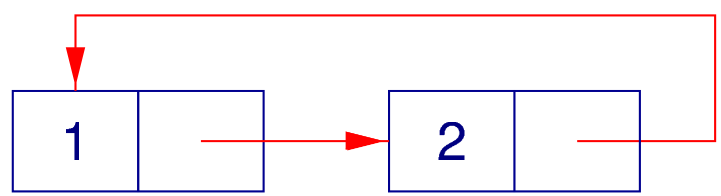
▷ Example

```
* RCons 1 (RCons 2 (Var Nothing)) :: CList Void
```



▷ `Var a` represents a backward pointer to an element in a list.

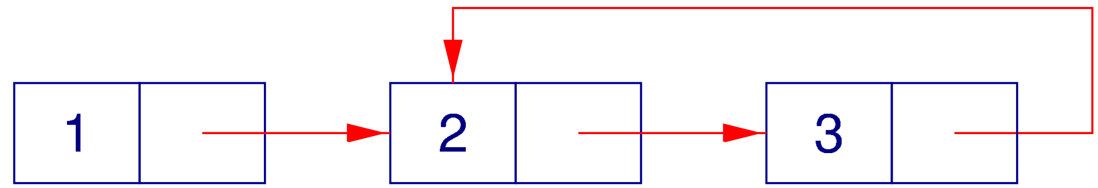
▷ `Nothing` is the pointer to the first element of a cyclic list.

▷ `Just Nothing` is for the second element, etc.

▷ The complete cyclic list has type `CList Void` (`Void` is def'd by `data Void`)

Examples

▷ `RCons 1 (RCons 2 (RCons 3 (Var (Just Nothing)))) :: CList Void`



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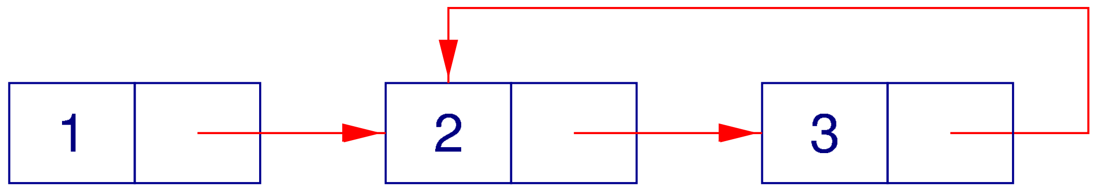

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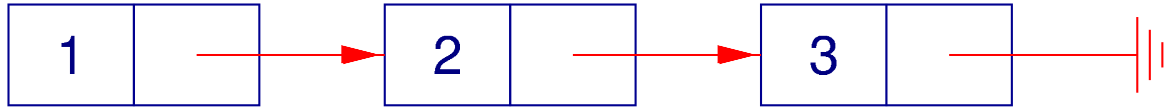

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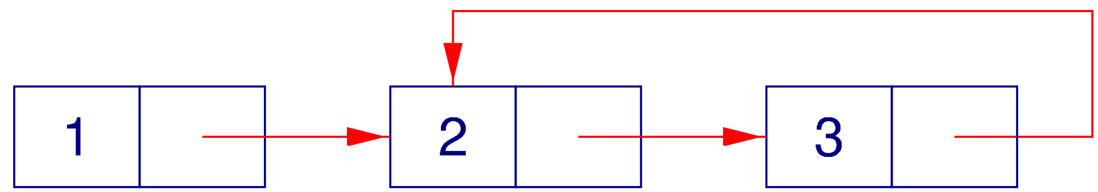

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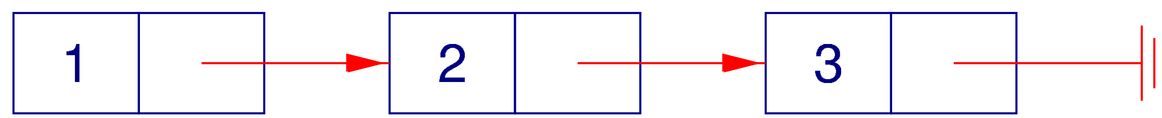

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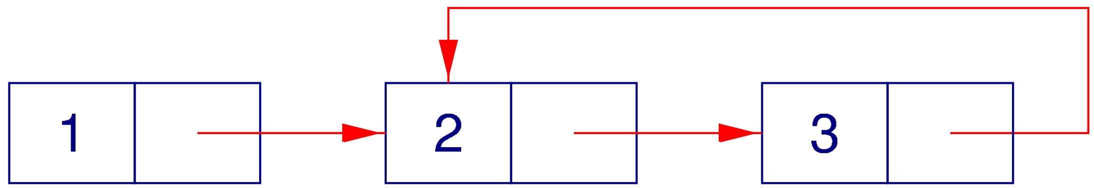

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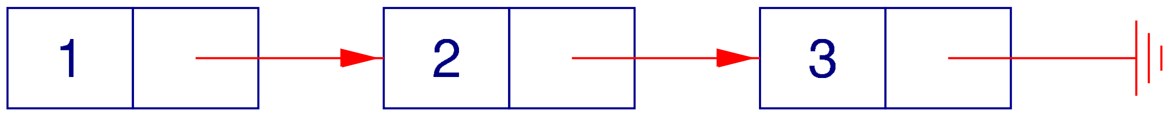

- ▷ Merit: no dangling pointer, i.e. no pointers which point outside the list
- ▷ If type `CList Void`, it is safe
- ▷ E.g. `(RCons 3 (Var (Just Nothing))) :: CList (Maybe (Maybe Void))`
- ▷ Different from integer pointer representation

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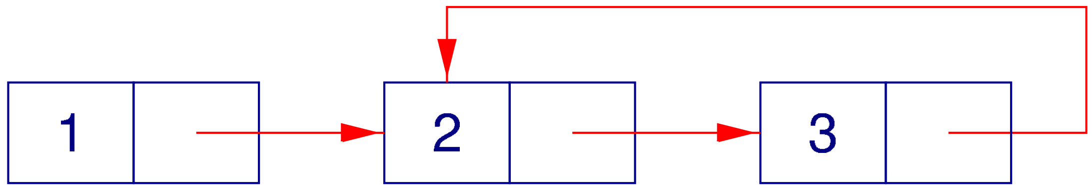

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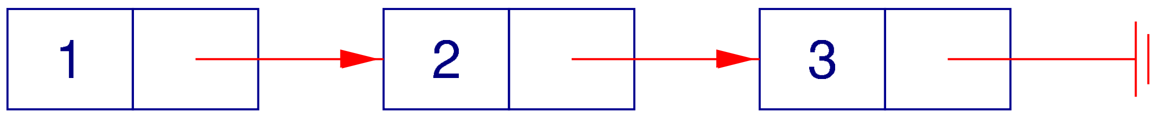

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▷ Different from integer pointer representation

▷ Unique representation

Plan

I. Main Part

- ▷ Cyclic lists
- ▷ Cyclic binary trees
- ▷ Semantics

II. More Details

- ▷ Generalized fold on cyclic lists
- ▷ General cyclic datatypes
- ▷ de Bruijn levels/indexes and type classes

I. Main Part

Cyclic Lists as Nested Datatype

```
data CList a = Var a
             | Nil
             | RCons Int (CList (Maybe a))
```

- ▷ List algebra structure on Cyclic Lists:

```
cnil :: CList Void          ccons :: Int -> CList Void -> CList Void
cnil = Nil                  ccons x xs = RCons x (shift xs)
```

```
shift :: CList a -> CList (Maybe a)
shift (Var z)        = Var (Just z)
shift Nil            = Nil
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- ▷ Since pointers denote "absolute positions", we need to `shift` the positions when consing $\Leftrightarrow$ de Bruijn's levels

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▷ If we use "relative positions" ($\Leftrightarrow$ de Bruijn's indexes) we don't need shifting $\cdots$ another problem

Cyclic Lists as Nested Datatype

- "Standard" fold:

```
cfold :: (forall a . a -> g a)
        -> (forall a . g a)
        -> (forall a . Int -> g (Maybe a) -> g a)
        -> CList a -> g a
cfold v n r (Var z)       = v z
cfold v n r Nil           = n
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- Example:

```
newtype K a = K Int
csum = cfold (\ x -> K 0) (K 0) (\ i (K j) -> K (i+j))
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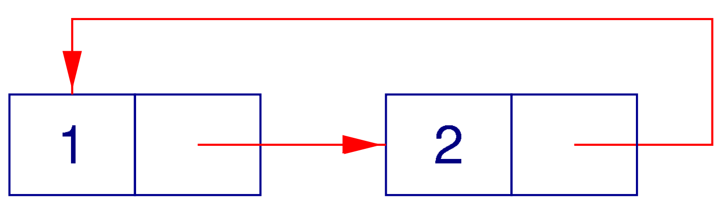
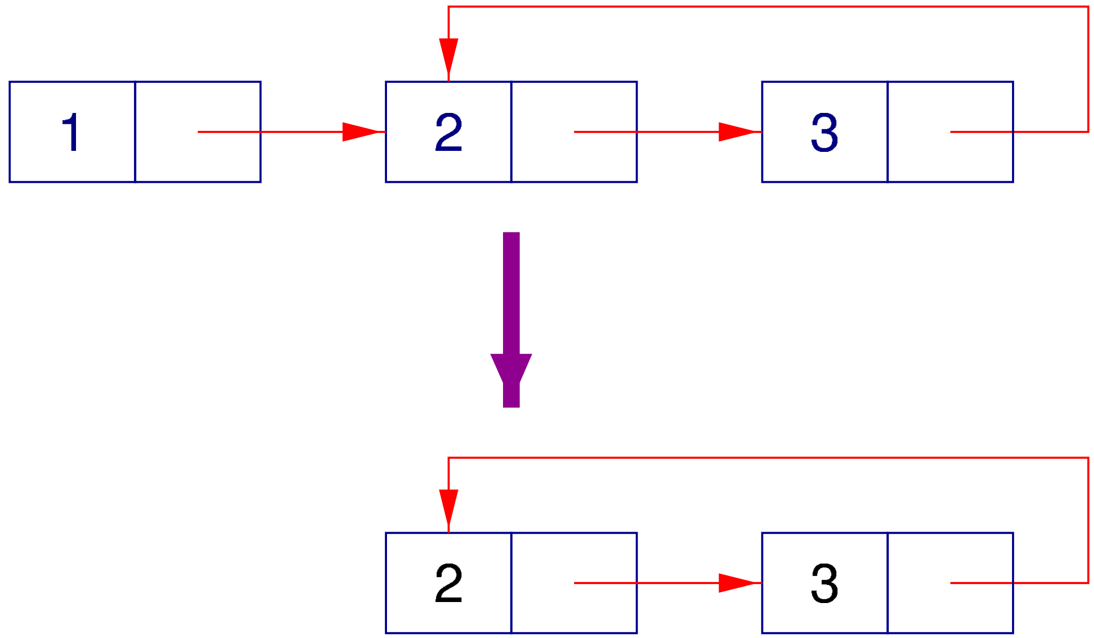
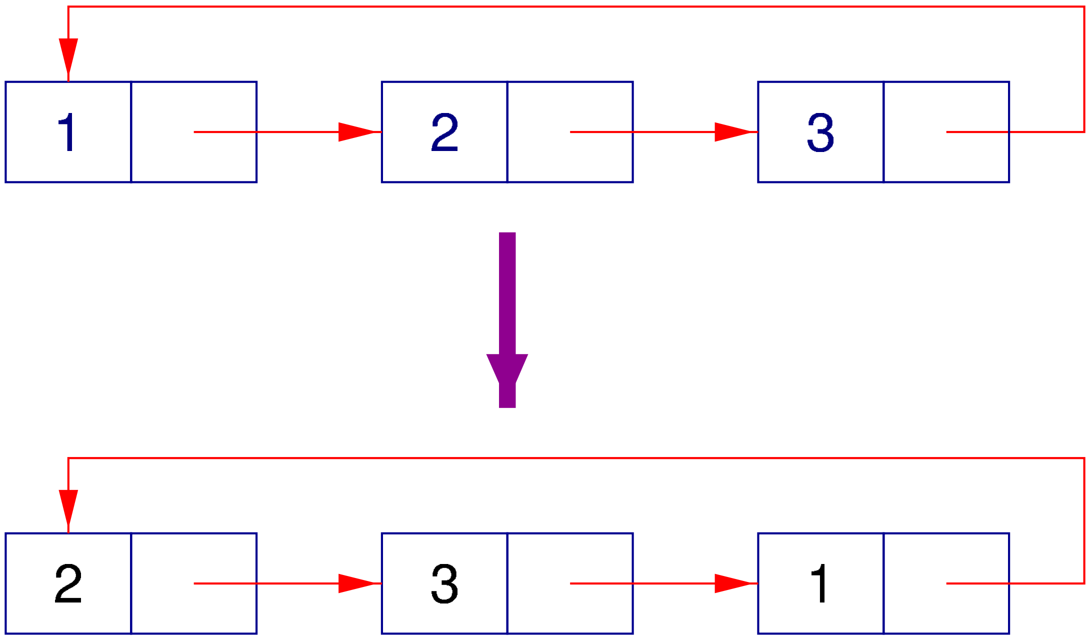



```
csum clist1   ==> 3
```

Cyclic Tail



Cyclic Tail – full cyclic case



▷ If the list is full cyclic, append the first element to the last,

▷ Otherwise, take a tail & decrease the pointer

Cyclic Lists as Nested Datatype

- ▷ List coalgebra structure on cyclic Lists:

```
chead :: CList Void -> Int
chead (RCons x _) = x

ctail :: CList Void -> CList Void
ctail (RCons x xs) = csnoc x xs
```

- ▷ `csnoc y xs` appends an element `y` to the last of `xs`

```
csnoc :: Int -> CList (Maybe a) -> CList a
csnoc y (Var Nothing)  = RCons y (Var Nothing)
csnoc y (Var (Just z)) = Var z
csnoc y Nil            = Nil
csnoc y (RCons x xs)   = RCons x (csnoc y xs)
```

Cyclic Lists as Nested Datatype

▷ Interpreting cyclic lists as infinite lists:

```
unwind :: CList Void -> [Int]
unwind Nil = []
unwind xs  = chead xs : unwind (ctail xs)
```

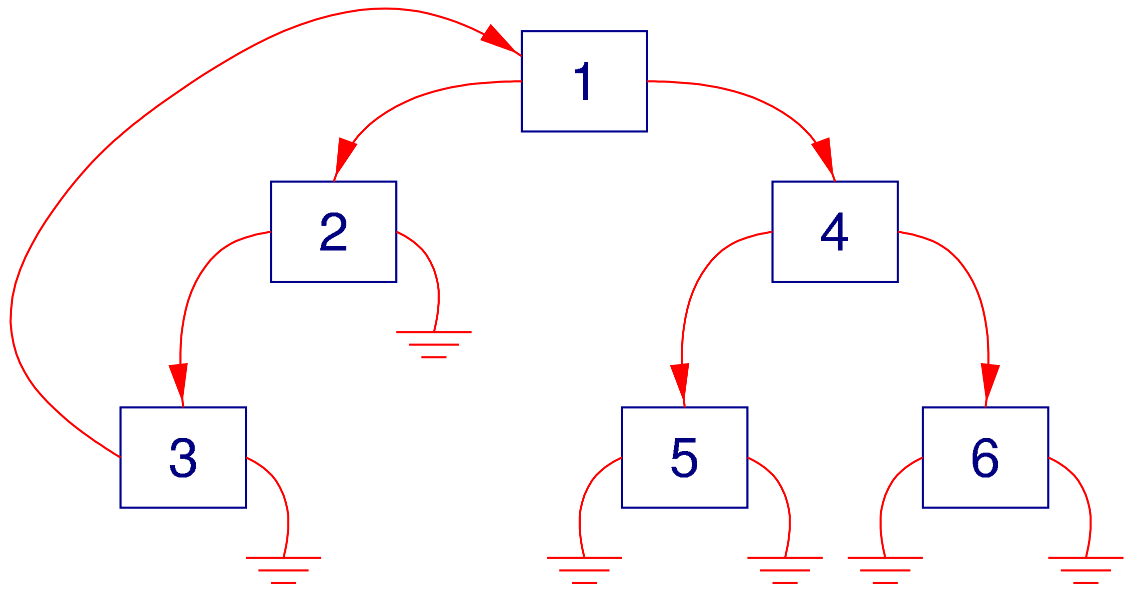
Cyclic Binary Trees

- Our Proposal of datatype of cyclic binary trees:

```
data CTree a = VarT a
             | Leaf
             | RBin Int (CTree (Maybe a))
                        (CTree (Maybe a))
```

- Cyclic binary trees with data at the nodes
- Each node has an "address" in top-down manner.
- All nodes on the same level have the same "address".
- Has only backpointers to form cycles.
- Pointers to other directions forbidden, hence no sharing.

Example




```
RBin 1 (RBin 2 (RBin 3 (VarT Nothing) Leaf)
                Leaf)
       (RBin 4 (RBin 5 Leaf Leaf)
               (RBin 6 Leaf Leaf))
```

Cyclic Binary Trees

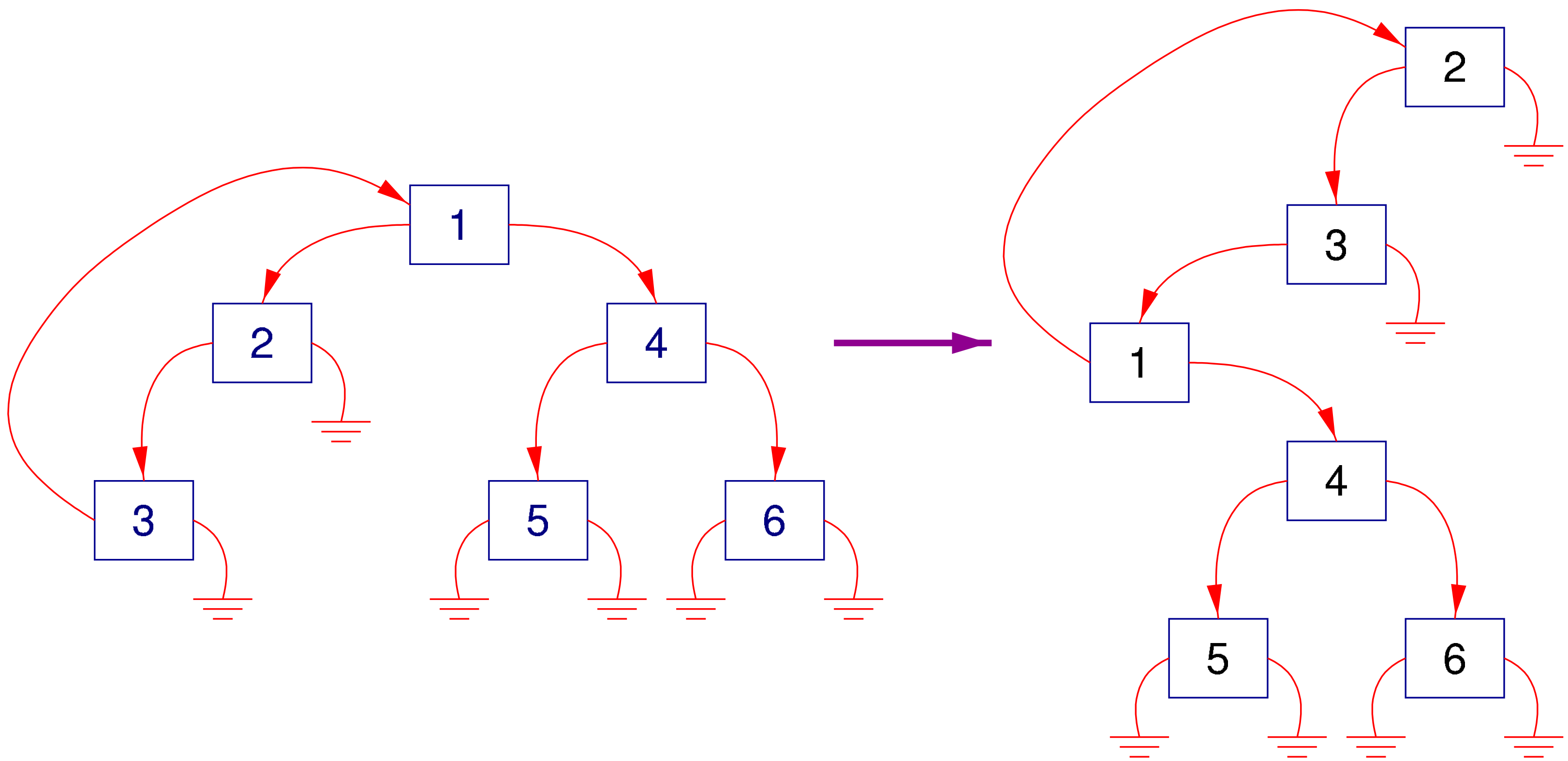
- Tree algebra structure:

```
cleaf :: CTree Void
cleaf = Leaf

cbin :: Int -> CTree Void -> CTree Void -> CTree Void
cbin x xsL xsR = RBin x (shiftT xsL) (shiftT xsR)

shiftT :: CTree a -> CTree (Maybe a)
shiftT (VarT z)          = VarT (Just z)
shiftT Leaf              = Leaf
shiftT (RBin x xsL xsR) = RBin x (shiftT xsL)
                                  (shiftT xsR)
```

Cyclic Children



Append "1" to the cyclic point
with keeping the right subtree

Cyclic Children

- Taking the left subtree operation:

```
csubL :: CTree Void -> CTree Void
csubL (RBin x xsL xsR) = csnocL x xsR xsL
```

- `csnocL y ys xs` appends an element `y` (with `ys`) to the leaf of `xs`

```
csnocL :: Int -> CTree (Maybe a)
              -> CTree (Maybe a) -> CTree a
csnocL y ys (VarT Nothing)   = RBin y (VarT Nothing) ys
csnocL y ys (VarT (Just z))  = VarT z
csnocL y ys Leaf             = Leaf
csnocL y ys (RBin x xsL xsR) = RBin y (csnocL y ys' xsL)
                                      (csnocL y ys' xsR)
                   where ys' = shiftT ys
```

- Generalization of `ctail`

Semantics – Cyclic Lists

```
data List = Nil | Cons Int List

data CList a = Var a
             | RNil
             | RCons Int (CList (Maybe a))

cnil                 = RNil
ccons x xs           = RCons x (shift xs)
chead (RCons x _)    = x
ctail (RCons x xs) = csnoc x xs
```

Semantics – Cyclic Lists

▷ List functor $\boldsymbol{F} : \mathbf{Set} \longrightarrow \mathbf{Set},$ $\boldsymbol{F}\boldsymbol{X} = \mathbf{1} + \mathbb{Z} \times \boldsymbol{X}$

Semantics – Cyclic Lists

- List functor $\boldsymbol{F} : \mathbf{Set} \longrightarrow \mathbf{Set}$, $\quad \boldsymbol{F}\boldsymbol{X} = \mathbf{1} + \mathbb{Z} \times \boldsymbol{X}$
- Cyclic list functor $\boldsymbol{G} : \mathbf{Set}^{\mathbf{Set}} \longrightarrow \mathbf{Set}^{\mathbf{Set}}$, $\quad \boldsymbol{G}\boldsymbol{A} = \mathrm{Id} + \mathbf{1} + \mathbb{Z} \times \boldsymbol{A}(\mathbf{1} + -)$

Semantics – Cyclic Lists

- ▷ List functor $F : \mathbf{Set} \to \mathbf{Set}$, $FX = 1 + \mathbb{Z} \times X$
- ▷ Cyclic list functor $G : \mathbf{Set}^{\mathbf{Set}} \to \mathbf{Set}^{\mathbf{Set}}$, $GA = \mathrm{Id} + 1 + \mathbb{Z} \times A(1 + -)$
- ▷ Initial G-algebra $GC \cong C \in \mathbf{Set}^{\mathbf{Set}}$
  $\mathbf{Set} \ni C_0 =$ `(CList Void)`

Semantics – Cyclic Lists

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- Initial G-algebra $GC \cong C \quad \in \mathbf{Set}^{\mathbf{Set}}$
  $\mathbf{Set} \ni C_0 = \texttt{(CList Void)}$

$$\begin{tikzcd}
\text{“finite lists”}\ F\mathbb{Z}^* \arrow[r, "\cong", "{[\texttt{nil}, \texttt{cons}]}"'] \arrow[d] & \mathbb{Z}^* \arrow[d] & \text{initial } F\text{-alg.} \qquad \text{in } \mathbf{Set} \\
FC_0 \arrow[r, "{[\texttt{cnil}, \texttt{ccons}]}"'] & C_0 \arrow[r, "\texttt{cpn}"'] \arrow[d] & FC_0 \arrow[d] \\
\text{“finite \& infinite lists”} & \mathbb{Z}^\infty \arrow[r, "\cong", "\texttt{pn}"'] & F\mathbb{Z}^\infty
\end{tikzcd}$$

“possible next”

$$\mathbb{Z}^\infty \qquad\qquad 1 + \mathbb{Z} \times \mathbb{Z}^\infty$$

$$\texttt{xs} \qquad\qquad * \text{ or } \langle \texttt{head}, \texttt{tail} \rangle (\texttt{xs})$$

final F-coalg.

II. More details

- ▷ Generalized fold
- ▷ General cyclic datatypes
- ▷ de Bruijn levels/indexes

Fold on Cyclic Lists

▷ "Standard" fold:

```
cfold :: (forall a . a -> g a)
        -> (forall a . g a)
        -> (forall a . Int -> g (Maybe a) -> g a)
        -> CList a -> g a
cfold v n r (Var z)       = v z
cfold v n r Nil           = n
cfold v n r (RCons x xs) = r x (cfold v n r xs)
```

▷ This gives `cfold (v n c) :: CList a -> T a`

Fold on Cyclic Lists

▷ General recursive definition

```
csnoc y (Var Nothing)  = RCons y (Var Nothing)
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- Instead: use `cfold (v n c) ::  CList a ->` $\boldsymbol{T}$ `a`

```
csnoc :: Int -> CList (Maybe a) -> CList a
csnoc z xs = cfold var Nil Cons xs
   where var Nothing  = RCons z (Var Nothing)
         var (Just n) = Var n
```

Fold on Cyclic Lists

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▷ Instead: use `cfold (v n c) :: CList a ->` $\boldsymbol{T}$ `a`

```
csnoc :: Int -> CList (Maybe a) -> CList a
csnoc z xs = cfold var Nil Cons xs
   where var Nothing  = RCons z (Var Nothing)
         var (Just n) = Var n
```

▷ But type mismatch!
Need: `cfold' (v n c) :: CList (Maybe a) ->` $\boldsymbol{T}$ `a`

Fold on Cyclic Lists

▷ Define `cfold' (v n c) :: CList (Maybe a) ->` $\boldsymbol{T}$ `a`

```
cfold' :: (forall a. Maybe a -> f a) ->
          (forall a . f a) ->
          (forall a. Int -> f (Maybe a) -> f a) ->
          CList (Maybe a) -> f a
cfold' v n c (Var x)    = v x
cfold' v n c Nil        = n
cfold' v n c (Cons x l) = c x (cfold' v n c l)
```

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```

▷ The same definition as "Standard" fold:

```
cfold :: (forall a . a -> g a)
        -> (forall a . g a)
        -> (forall a . Int -> g (Maybe a) -> g a)
        -> CList a -> g a
cfold v n r (Var z)      = v z
cfold v n r Nil          = n
cfold v n r (RCons x xs) = r x (cfold v n r xs)
```

Fold on Cyclic Lists

- Generalized fold for nested datatype via a right Kan extension:

  `cefold (v n c) ::  CList (`$\boldsymbol{M}$` a) -> `$\boldsymbol{T}$` a`

  [Bird,Paterson'99][Martin,Gibbons,Bayley'04][Abel,Matthes,Uustalu'05]

```
cefold :: (forall a. Maybe (m a) -> h (Maybe a))
     -> (forall a . m a -> t a)
     -> (forall a . t a)
     -> (forall a . Int -> g (Maybe a) -> t a)
     -> CList (m a) -> t a
cefold d v n r (Var z)       = v z
cefold d v n r Nil           = n
cefold d v n r (RCons x xs) = r x (cefold d v n r (fmap d xs))
```

- `d` is a distributive law.

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For any given algebraic datatype, we can give its cyclic version.

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- Lists

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- Derivative of datatype is useful

General Cyclic Datatypes

▷ Binary trees

$$\boldsymbol{FX = 1 + \mathbb{Z} \times X \times X}$$

▷ Derivative of datatype (e.g. binary trees, $\boldsymbol{(1 + zx^2)' = 2zx}$)
$\boldsymbol{F'X = \mathbb{Z}X + \mathbb{Z}X}$ gives a "one-hole context" [McBride'01]

General Cyclic Datatypes

- Binary trees

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- Derivative of datatype (e.g. binary trees, $\boldsymbol{(1 + z x^2)' = 2 z x}$)

  $\boldsymbol{F' X = \mathbb{Z} X + \mathbb{Z} X}$ gives a "one-hole context" [McBride'01]

- Original snoc for binary trees

```
csnocL :: Int -> CTree (Maybe a)
              -> CTree (Maybe a) -> CTree a
csnocL y ys (VarT Nothing) = RBin y (VarT Nothing) ys
...
```

- One-hole context is useful:

  `combCtx ::` $\boldsymbol{F' X \times X \longrightarrow F X}$ is the "plug-in" operation that fills a hole

```
csnocL ctx (VarT Nothing) = combCtx ctx (VarT Nothing)
```

Conclusions

▷ Generic framework to model cyclic structures

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To do:

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- ▷ Develop a categorical account of rational and cyclic coinductive types
- ▷ Practical examples
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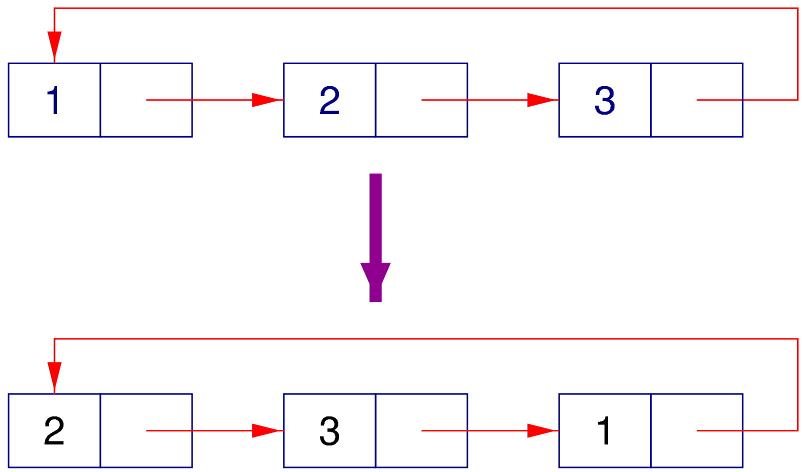
Paper, slides and programs at:

`http://www.keim.cs.gunma-u.ac.jp/~hamana/`

De Bruijn Indexes

- ▷ Relative pointers rather than absolute ones
  - * Relative: `RCons 1 (RCons 2 (Var (Just (Just Nothing))))`



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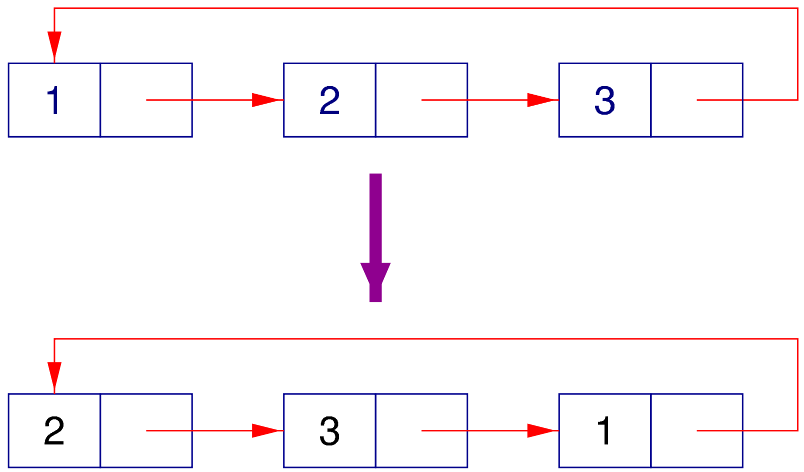

- ▷ This case:

```
ccons :: Int -> CList Void -> CList Void
ccons x xs = RCons x xs
```

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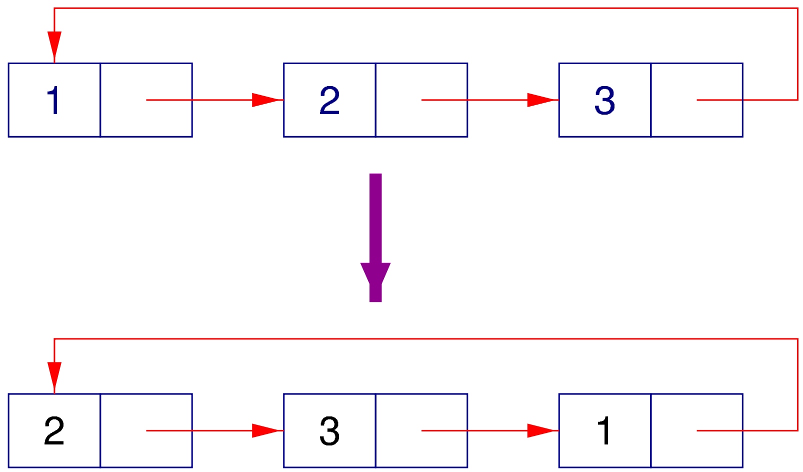

- ▷ This case:

```
ccons :: Int -> CList Void -> CList Void
ccons x xs = RCons x xs
```

```
-- RCons :: Int -> CList (Maybe a) -> CList a
```

But type mismatch

De Bruijn Indexes

▷ Second try:

```
ccons :: Int -> CList Void -> CList Void
ccons x xs = RCons x (emb xs)

emb :: CList a -> CList (Maybe a)
emb (Var z)       = Var z
emb Nil           = Nil
emb (RCons x xs) = RCons x (emb xs)
```

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▷ Second try:

```
ccons :: Int -> CList Void -> CList Void
ccons x xs = RCons x (emb xs)

emb :: CList a -> CList (Maybe a)
emb (Var z)       = Var z
emb Nil           = Nil
emb (RCons x xs) = RCons x (emb xs)
```

```
ERROR "clists.hs":36 - Type error in explicitly typed binding
*** Term           : emb
*** Type           : CList a -> CList a
*** Does not match : CList a -> CList (Maybe a)
*** Because        : unification would give infinite type
```

De Bruijn Indexes – Correct Definition

```
class DeBrIdx a where
  wk  :: a -> Maybe a
```

```
instance DeBrIdx Void where
  wk _ = undefined
```

```
instance DeBrIdx a => DeBrIdx (Maybe a) where
  wk Nothing = Nothing
  wk (Just x) = Just (wk x)

instance Functor CList where
  fmap f (Var a) = Var (f a)
  fmap f Nil     = Nil
  fmap f (RCons x xs) = RCons x (fmap (fmap f) xs)

emb :: DeBrIdx a => CList a -> CList (Maybe a)
emb = fmap wk

ccons :: Int -> CList Void -> CList Void
ccons x xs = RCons x (emb xs)
```

De Bruijn Indexes – Correct Definition

```
emb :: DeBrIdx a => CList a -> CList (Maybe a)
ccons :: Int -> CList Void -> CList Void
ccons x xs = RCons x (emb xs)
```

- ▷ Typed program is less efficient than untyped program?
- ▷ Type equality coercion? (suggested by Simon Peyton-Jones at TFP’06)
  Core language of Haskell: System F with type equality coercion