Universal Algebra for Termination of Higher-Order Rewriting

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Intro: First-Order Term Rewriting System (TRS)

- ▷ Terms $T_\Sigma X \ni t ::= x \mid f(t_1, \ldots, t_m)$
- ▷ Rewrite rules $\mathcal{R} = \{t_1 \to t_2, \ldots\}$
- ▷ Rewrite relation $\to_{\mathcal{R}}$
- ▷ An important problem: termination of $\mathcal{R}$

Intro: Complete Algebraic Characterisation of TRSs

Theorem [Lankford'79, Zantema'94]

A TRS $\mathcal{R}$ is terminating if and only if there exists a well-founded monotone $(\Sigma, \mathcal{R})$-algebra.

("ordered" Σ-algebra $(A, >_A)$ that validates all rules in $\mathcal{R}$)

I. Useful: Finding a well-founded $(\Sigma, \mathcal{R})$-algebra implies termination of $\mathcal{R}$

II. Fundamental: Sound and complete semantics of TRS

III. Applications: Semantic labeling, monadic modularity, etc.

Q. Does a similar theorem hold for higher-order* rewriting?

* Rewriting on terms with variable binding and (meta-level) substitutions

A General Church-Rosser Theorem
Aczel (1978)
(unpublished manuscript)

Binding Algebras:
A Step between
Universal Algebra and Type Theory
Plotkin (1998)
(RTA'98, Tsukuba, Japan)

Combinatory Reduction Systems
Klop (1980)
(PhD thesis)

Abstract Syntax and Variable Binding
Fiore, Plotkin, Turi (LICS'99)

(Works on CRSs around 90s)
Klop, Oostrom,
Raamsdonk (TCS'93)

This work (RTA'05)
Universal Algebra for Termination of Higher-Order Rewriting

Contents

Algebraic semantics for

I. CRS rewriting

II. CRS meta-rewriting

III. Binding CRSs

Combinatory Reduction System (CRS) [Klop'80]

Example: conversion into prenex normal form

$$
\begin{array}{llll}
\mathrm{P} \wedge \forall(x.\mathrm{Q}[x]) & \to \forall(x.\mathrm{P} \wedge \mathrm{Q}[x]) & \neg\forall(x.\mathrm{Q}[x]) & \to \exists(x.\neg(\mathrm{Q}[x])) \\
\forall(x.\mathrm{Q}[x]) \wedge \mathrm{P} & \to \forall(x.\mathrm{P} \wedge \mathrm{Q}[x]) & \neg\exists(x.\mathrm{Q}[x]) & \to \forall(x.\neg(\mathrm{Q}[x]))
\end{array}
$$

Definition

Variables	$x, y, z, \cdots$
Metavariables	Z^l (arity l), $\cdots$
Function symbols	F^l (arity l), $\cdots \in \Sigma$
Terms	$s ::= x \mid x.s \mid F^l(s_1, \ldots, s_l)$
Meta-terms	$t ::= x \mid x.t \mid F^l(t_1, \ldots, t_l) \mid \mathrm{Z}^l\,[t_1, \ldots, t_l]$
CRS rules $\mathcal{R}$	$t_1 \to t_2$ (with some syntactic conditions)
CRS rewriting $\to_{\mathcal{R}}$	

$$
\frac{l \to r \in \mathcal{R}}{\theta(l) \to_{\mathcal{R}} \theta(r)} \qquad \frac{s \to_{\mathcal{R}} t}{x.s \to_{\mathcal{R}} x.t} \qquad \frac{s \to_{\mathcal{R}} t}{F(\ldots, s, \ldots) \to_{\mathcal{R}} F(\ldots, t, \ldots)}
$$

A valuation θ is a mapping from metavariables to terms.

Technical Framework

Algebras in the functor category $\mathbf{Set}^{\mathbb{F}}$
(first-order case: algebras are in $\mathbf{Set}$)

$\mathbb{F}$ $\cdots$ the category of natural numbers and all functions

A presheaf $X \in \mathbf{Set}^{\mathbb{F}}$ is an $\mathbb{N}$-indexed set with arrow part.

Definition [Fiore,Plotkin,Turi]

Σ $\cdots$ the functor $\Sigma : \mathbf{Set}^{\mathbb{F}} \longrightarrow \mathbf{Set}^{\mathbb{F}}$ given by a binding signature

A Σ-monoid consists of

- a monoid object A in the monoidal category $(\mathbf{Set}^{\mathbb{F}}, \bullet, \mathbf{V})$ with
- a Σ-algebra $\alpha : \Sigma A \longrightarrow A$

such that "the monoid multiplication and α commutes".

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Definition [Fiore,Plotkin,Turi]

$\boldsymbol{\Sigma} \cdots$ the functor $\boldsymbol{\Sigma} : \mathbf{Set}^{\mathbb{F}} \longrightarrow \mathbf{Set}^{\mathbb{F}}$ given by a binding signature

A $\boldsymbol{\Sigma}$-monoid consists of

- a monoid object = modelling substitutions
- a $\boldsymbol{\Sigma}$-algebra = modelling $\boldsymbol{\Sigma}$-terms

such that "substitution on $\boldsymbol{\Sigma}$-term" is defined.

Structural Terms

Observation

Syntax given by the free Σ-monoid $M_\Sigma\hat{Z}$ over $\hat{Z}$ [Hamana APLAS'04]

$$M_\Sigma\hat{Z}(n) \ni t ::= x \mid F^l(n{+}1,\ldots,n{+}i.t_1,\ldots,n{+}1,\ldots,n{+}l.t_l) \mid \mathrm{z}^l\,[t_1,\ldots,t_l]$$

▶ Similarity to CRS Meta-terms

$$t ::= x \mid x.t \mid F^l(t_1,\ldots,t_l) \mid \mathrm{z}^l\,[t_1,\ldots,t_l]$$

Binding signature Σ

$F : \langle n_1,\ldots,n_l\rangle \in \Sigma \qquad (1 \le i \le l)$

F has l arguments and binds n_i variables in the i-th argument.

▶ Define Structural meta-terms by
meta-terms built from a binding signature.

▶ Then, structural meta-terms forms a free Σ-monoid.

▶ Structural meta-terms have a good structural induction principle

(due to the initial algebra property).

Algebraic Semantics of CRS Syntax

By [Fiore,Plotkin,Turi'99] [Hamana'04],

Thm. Structural CRS terms $T_\Sigma \mathbf{V}$ forms an initial $\mathbf{V}+\Sigma$-algebra.

Thm. Structural CRS meta-terms $M_\Sigma Z$ forms a free Σ-monoid over $\hat{Z}$.

Assumption: Use the method of de Bruijn levels for (meta-)terms

- CRS's variables are $\mathbf{1}, \mathbf{2}, \mathbf{3}, \cdots \in \mathbb{N}$.

Presheaves: Use structural (mata-)term sets parameterised by the set of free variables $n = \{1, \ldots, n\}$.

Define

$$\mathbf{V}(n) \triangleq \{1, \ldots, n\} \;=\; n \quad \text{“vars from } 1 \text{ to } n\text{”}$$

$$T_\Sigma \mathbf{V}(n) \triangleq \{t \mid \text{term } t \text{ has } n\text{-free vars}\}$$

$$M_\Sigma Z(n) \triangleq \{t \mid \text{meta-term } t \text{ has } n\text{-free vars}\}$$

$$Z(l) \triangleq \{\mathrm{z} \mid \text{metavariable } \mathrm{z} \text{ has arity } l\}$$

Then, $\mathbf{V}, \hat{Z}, T_\Sigma \mathbf{V}, M_\Sigma Z \in \mathbf{Set}^{\mathbb{F}}$.

Structural CRSs

Definition

A Structural CRS is a CRS built from structural meta-terms.

Similarly for other syntactic objects: valuation, rewrite relation.

Assumption:

Hereafter, we only consider structural CRSs.

We just say "a CRS" for a structural CRS.

Algebraic Semantics of CRS rewriting

Example: $Z = \{\mathrm{M}^1, \mathrm{N}^0\}$ CRS for untyped λ-calculus

Rule app(lam$(1.\mathrm{M}), \mathrm{N}) \longrightarrow \mathrm{M}[\mathrm{N}]$

$$\frac{l \rightarrow r \in \mathcal{R}}{\theta(l) \rightarrow_{\mathcal{R}} \theta(r)}$$

▷ Valuation $\theta : Z \longrightarrow T_\Sigma\mathrm{V}$ into terms.

$$\mathrm{M}^1 \longmapsto t \in T_\Sigma\mathrm{V}(1) \qquad \text{“}t \text{ has at most 1-free var”}$$

$$\mathrm{N}^0 \longmapsto s \in T_\Sigma\mathrm{V}(0) \qquad \text{“}s \text{ has no free vars”}$$

▶ A valuation is characterised as a morphism of $\mathbf{Set}^{\mathbb{F}}$.

▷ How to interpret rewrite rules?

case for TRS:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\phi"'] & T_\Sigma(X) \arrow[d, dotted, "\phi^*"] \\ \Sigma\text{-algebra} & A \end{tikzcd} \qquad \begin{tikzcd} l \rightarrow r \in \mathcal{R} \arrow[d, mapsto] \\ \phi^*(l) >_A \phi^*(r) \end{tikzcd}$$

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$l \to r \in \mathcal{R}$

?

$\phi^*(l) >_A \phi^*(l)$

Algebraic semantics of CRS rewriting

$$
\begin{tikzcd}
& Z \arrow[r, "\eta_Z"] \arrow[dr, "\theta"'] & M_{\Sigma} Z \arrow[d, "\theta^*"] & Z|n \vdash l \to r \in \mathcal{R} & \\
& & & \text{unique } \Sigma\text{-monoid mor.} & \\
\Sigma\text{-monoid} & & T_{\Sigma}\mathrm{V} \arrow[d, "!_A"] & \theta^*_n(l) \to_{\mathcal{R}} \theta^*_n(r) & s \to^+_{\mathcal{R}} t \arrow[d, mapsto] \\
\left(\begin{array}{c} =\text{initial} \\ \mathrm{V}+\Sigma\text{-algebra} \end{array} \right) & & & \text{unique } \mathrm{V}+\Sigma\text{-algebra hom.} & \\
\mathrm{V}+\Sigma\text{-algebra} & & A & !_A\theta^*_n(l) >_{A(n)} !_A\theta^*_n(l) & !_{A(n)}(s) >_{A(n)} !_{A(n)}(t)
\end{tikzcd}
$$

▶ $(\mathrm{V}+\Sigma, \mathcal{R})$-algebra

Theorem *For any $(\mathrm{V}+\Sigma, \mathcal{R})$-algebra A, there exists a unique monotone homomorphism* $(T_{\Sigma}\mathrm{V}, \to^+_{\mathcal{R}}) \longrightarrow (A, >_A)$.

Algebraic semantics of CRS rewriting

Main Theorem 1 (Complete characterisation of termination)

A CRS $\mathcal{R}$ is terminating if and only if there exists a well-founded $(\mathbf{V}+\mathbf{\Sigma},\mathcal{R})$-algebra.

Meta-rewriting

More direct interpretation is possible.

- Rewriting on Meta-terms = Meta-rewriting

CRS $(\mathcal{R}, Z)$

$$\frac{\vec{n}.l \to \vec{n}.r \in \mathcal{R}}{n \vdash \theta^*(n)(l) \rightsquigarrow_{\mathcal{R}} \theta^*(n)(l)} \qquad \frac{n{+}i \vdash s \rightsquigarrow_{\mathcal{R}} t}{n \vdash F(\ldots, n{+}\vec{i}.s, \ldots) \rightsquigarrow_{\mathcal{R}} F(\ldots, n{+}\vec{i}.t, \ldots)}$$

$$\frac{\mathrm{Z} \in Z(l) \quad n \vdash s \rightsquigarrow_{\mathcal{R}} t}{n \vdash \mathrm{Z}[\ldots, s, \ldots] \rightsquigarrow_{\mathcal{R}} \mathrm{Z}[\ldots, t, \ldots]}$$

θ is a mapping $Z \longrightarrow M_{\Sigma}X$ to meta-terms

Algebraic Semantics of Meta-rewriting

Theorem $\forall\, A$: $(\Sigma, \mathcal{R})$-monoid

$$\begin{tikzcd} Z \arrow[r, "\eta_Z"] \arrow[dr, "\phi"', "\text{admissible}"'] & M_\Sigma Z \arrow[d, dotted, "\phi^* \ \exists!\ \text{monotone } \Sigma\text{-monoid mor.}"] \\ & A \end{tikzcd}$$

$$n \vdash s \rightsquigarrow^{+}_{\mathcal{R}} t$$

$$\phi(n)^*(s) >_{A(n)} \phi(n)^*(t)$$

Main Theorem 2 (Complete characterisation of meta-termination)

A CRS $(\mathcal{R}, Z)$ is meta-terminating if and only if there exists a well-founded $(\Sigma, \mathcal{R})$-monoid.

Binding CRSs

(Essentially) meta-application free fragment

- Simpler interpretation not using monoid structure
- Example:

$$
\begin{array}{llll}
\mathbf{P} \wedge \forall(\boldsymbol{x}.\mathbf{Q}[\boldsymbol{x}]) & \longrightarrow \forall(\boldsymbol{x}.\mathbf{P} \wedge \mathbf{Q}[\boldsymbol{x}]) & \neg\forall(\boldsymbol{x}.\mathbf{Q}[\boldsymbol{x}]) & \longrightarrow \exists(\boldsymbol{x}.\neg(\mathbf{Q}[\boldsymbol{x}])) \\
\forall(\boldsymbol{x}.\mathbf{Q}[\boldsymbol{x}]) \wedge \mathbf{P} & \longrightarrow \forall(\boldsymbol{x}.\mathbf{P} \wedge \mathbf{Q}[\boldsymbol{x}]) & \neg\exists(\boldsymbol{x}.\mathbf{Q}[\boldsymbol{x}]) & \longrightarrow \forall(\boldsymbol{x}.\neg(\mathbf{Q}[\boldsymbol{x}]))
\end{array}
$$

- Crucial fact: for the presheaves $\boldsymbol{Z}$:metavariables, $\mathbf{V}$:vars,

  $\mathbf{V}$ *is the unit of the monoidal category* $(\mathbf{Set}^{\mathbb{F}}, \bullet, \mathbf{V})$.

Binding CRSs

(Essentially) meta-application free fragment

- ▷ Simpler interpretation not using monoid structure
- ▷ Example:

$$
\begin{array}{llll}
\mathrm{P} \wedge \forall(x.\mathrm{Q}[x]) & \to \forall(x.\mathrm{P} \wedge \mathrm{Q}[x]) & \neg\forall(x.\mathrm{Q}[x]) & \to \exists(x.\neg(\mathrm{Q}[x])) \\
\forall(x.\mathrm{Q}[x]) \wedge \mathrm{P} & \to \forall(x.\mathrm{P} \wedge \mathrm{Q}[x]) & \neg\exists(x.\mathrm{Q}[x]) & \to \forall(x.\neg(\mathrm{Q}[x]))
\end{array}
$$

- ▷ Crucial fact: for the presheaves $\boldsymbol{Z}$:metavariables, $\mathbf{V}$:vars,

  $\boldsymbol{Z} \bullet \mathbf{V} \cong \boldsymbol{Z}$ e.g. $\boldsymbol{Z} \bullet \mathbf{V}(1) \ni \mathrm{Q}[x] = \mathrm{Q} \in \boldsymbol{Z}(1)$

- ▷ Binding CRSs $\overset{def}{\Longleftrightarrow}$ CRSs consisting of binding meta-terms

  ($\boldsymbol{Z}, \mathbf{V}$ and $\boldsymbol{\Sigma}$-terms)

  $t ::= x \mid F(x_1 \cdots x_{i_1}.t_1, \ldots, x_1 \cdots x_{i_l}.t_l) \mid \mathrm{Z}^l \qquad (= \mathrm{Z}^l\,[1, \ldots, l])$

- ▷ Consider $\boldsymbol{Z}+\mathbf{V}+\boldsymbol{\Sigma}$-algebras

  Binding meta-terms $=$ the initial $\boldsymbol{Z}+\mathbf{V}+\boldsymbol{\Sigma}$-algebra

Algebraic Semantics of Binding CRSs

- Redo everything for binding CRSs. We again obtain:

- **Proposition**

  A binding CRS $(\mathcal{R}, Z)$ is meta-terminating on all binding meta-terms if and only if there exists a well-founded $(Z + \mathbf{V} + \mathbf{\Sigma}, \mathcal{R})$-algebra.

- This gives a simpler interpretation method of showing termination of binding CRSs:
  because $\{$binding meta-terms$\} \supseteq \{$terms$\}$.

Binding CRSs – Termination Proof

Example

$$\mathbf{P} \wedge \forall(x.\mathbf{Q}[x]) \;\to\; \forall(x.\mathbf{P} \wedge \mathbf{Q}[x]) \qquad \neg\forall(x.\mathbf{Q}[x]) \;\to\; \exists(x.\neg(\mathbf{Q}[x]))$$

$$\forall(x.\mathbf{Q}[x]) \wedge \mathbf{P} \;\to\; \forall(x.\mathbf{P} \wedge \mathbf{Q}[x]) \qquad \neg\exists(x.\mathbf{Q}[x]) \;\to\; \forall(x.\neg(\mathbf{Q}[x]))$$

$Z + \mathbf{V} + \boldsymbol{\Sigma}$-algebra K:

Carrier: $K(n) = \mathbb{N}$ with the usual order

Operations:

$$\wedge_{K(n)}(x, y) = \vee_{K(n)}(x, y) = 2x + 2y$$

$$\neg_{K(n)}(x) = 2x \qquad \forall_{K(n)}(x) = \exists_{K(n)}(x) = x + 1$$

$Z + \mathbf{V} + \boldsymbol{\Sigma}$-algebra K satisfies $\mathcal{R}$. i.e. $(Z + \mathbf{V} + \boldsymbol{\Sigma}, \mathcal{R})$-algebra.

Hence $\mathcal{R}$ is terminating on all CRS terms.

▶ Simpler than existing proof methods of termination of higher-order rewriting [van de Pol'96].

Summary

I. Universal Algebra for CRSs = $\mathbf{\Sigma}$-monoids

II. Complete characterisations of termination and meta-termination of CRSs

III. Universal Algebra for binding CRSs = $\boldsymbol{Z}+\mathbf{V}+\mathbf{\Sigma}$-algebras

- Simpler interpretation for termination proof

Thank you.