Polymorphic Abstract Syntax via Grothendieck Construction

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This Work

The initial algebra characterisation of
polymorphic typed abstract syntax with variable binding

(1) Second-order polymorphic syntax:
e.g. System $\mathbf{F}$, and variants

(2) Higher-order polymorphic syntax:
e.g. System $\mathbf{F}_\omega$, and variants

History: Initial Algebra Semantics of Syntax

- Abstract syntax is formulated as an *initial algebra*

FO	Abstract syntax	**Set**	ADJ	1975

History: Initial Algebra Semantics of Syntax

▷ *Enriched* abstract syntax are formulated as *initial algebras* in various presheaf categories

FO Abstract syntax	$\mathbf{Set}$	ADJ	1975
Abstract syntax with binding	$\mathbf{Set}^{\mathbb{F}}$	Fiore,Plotkin,Turi Hofmann	1999
Simply-typed syn. with binding	$(\mathbf{Set}^{\mathbb{F}\downarrow U})^{U}$	Fiore	2003
Dependently-sorted syntax	$(\mathbf{Set}^{\mathbf{Fin}[\mathbb{S},\mathbf{Set}]})^{\mathbb{S}}$	Fiore	2008

Initial Algebra Semantics of Abstract Syntax

▷ Algebraic characterisations have produced fruitful applications

Abstract syntax with binding	$\mathbf{Set}^{\mathbb{F}}$	Fiore,Plotkin,Turi	1999
- with metavariables		Hamana,Fiore	2004
- Explicit substitutions		Ghani,Uustalu,Hamana	2006
- Fusion calculus		Miculan	2008
Simply-typed syn. with binding	$(\mathbf{Set}^{\mathbb{F}\downarrow U})^{U}$	Fiore	2003
- Normalisation by evaluation			
- Pre-logical predicates		Katsumata	2004
- Cyclic sharing tree structures		Hamana	2009
- Second-order equational logic		Fiore,Hur	2010

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Polymorphic Abstract Syntax	?	?	?
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Polymorphic Abstract Syntax	$\mathbf{Set}^{\int \mathbf{G}}$	Hamana	2011
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I. Abstract Syntax with Binding

- Aim: To model syntax with variable binding, e.g.

$$\frac{}{x_1, \ldots, x_n \vdash x_i} \qquad \frac{x_1, \ldots, x_n \vdash t \quad x_1, \ldots, x_n \vdash s}{x_1, \ldots, x_n \vdash t@s}$$

$$\frac{x_1, \ldots, x_n, x_{n+1} \vdash t}{x_1, \ldots, x_n \vdash \lambda(x_{n+1}.t)}$$

- Syntax generated by 3 constructors
- λ is a special unary function symbol: decreases the context

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- Category $\mathbb{F}$ for contexts

  objects: $n = \{1, \ldots, n\}$ (contexts)

  arrows: all functions $n \longrightarrow n'$ (renamings)

- Endofunctor $\Sigma_\lambda : \mathbf{Set}^{\mathbb{F}} \longrightarrow \mathbf{Set}^{\mathbb{F}}$

$$\Sigma_\lambda(A) = \mathbf{V} + A \times A + \delta A$$

Context extension $\delta A(n) = A(n+1)$

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- Thm. Initial Σ_λ-algebra characterises λ-terms

II. Typed Abstract Syntax with Binding

▷ Aim: To model simply-typed syntax with variable binding, e.g.

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma \vdash t : \sigma \Rightarrow \tau \quad \Gamma \vdash s : \sigma}{\Gamma \vdash t@s : \tau}$$

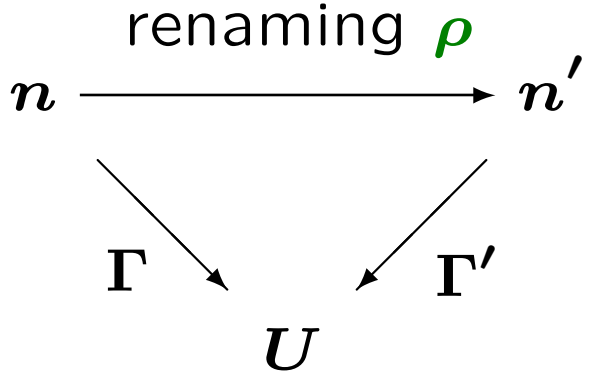
$$\frac{\Gamma, x : \sigma \vdash t : \tau}{\Gamma \vdash \lambda(x{:}\sigma.t) : \sigma \Rightarrow \tau}$$

▷ Need: typed contexts $\Gamma = \{x_1 : \tau_1, \cdots, x_n : \tau_n\}$

▷ Category $(\mathbb{F} \downarrow U)$ [Fiore'02, Miculan, Scagnetto'03]

objects: $\Gamma : n \longrightarrow U$ (contexts)

arrows:



U: set of all types

"preserves types"

II. Typed Abstract Syntax with Binding

$$\frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma \vdash t : \sigma \Rightarrow \tau \quad \Gamma \vdash s : \sigma}{\Gamma \vdash t@s : \tau}$$

$$\frac{\Gamma, x : \sigma \vdash t : \tau}{\Gamma \vdash \lambda(x{:}\sigma.t) : \sigma \Rightarrow \tau}$$

- ▷ Category of discourse: $(\mathbf{Set}^{\mathbb{F}\downarrow U})^U$
  - $\mathbb{F}\downarrow U$: contexts Γ
  - U: target types τ
- ▷ Endofunctor $\Sigma_{\vec{\lambda}} : (\mathbf{Set}^{\mathbb{F}\downarrow U})^U \longrightarrow (\mathbf{Set}^{\mathbb{F}\downarrow U})^U$
- ▷ Thm. Initial algebra characterises well-typed terms

Second-order Polymorphic Abstract Syntax

Examples: System F, variants of F, etc.

System F

Types

$$\tau ::= \alpha \mid b \mid \tau_1 \Rightarrow \tau_2 \mid \forall \alpha.\tau$$

$\alpha \cdots$ type variables $\quad b \cdots$ base types

Well-formed types

$$\frac{1 \leq i \leq n}{\alpha_1, \ldots, \alpha_n \vdash \alpha_i} \qquad \frac{}{\alpha_1, \ldots, \alpha_n \vdash b}$$

$$\frac{\alpha_1, \ldots, \alpha_n \vdash \sigma \quad \alpha_1, \ldots, \alpha_n \vdash \tau}{\alpha_1, \ldots, \alpha_n \vdash \sigma \Rightarrow \tau} \qquad \frac{\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \vdash \tau}{\alpha_1, \ldots, \alpha_n \vdash \forall \alpha_{n+1}.\tau}$$

Well-typed terms

$$\frac{\Xi, \alpha \mid \Gamma \vdash t : \tau}{\Xi \mid \Gamma \vdash \Lambda\alpha.\, t : \forall\alpha.\, \tau} \qquad \frac{\Xi \mid \Gamma \vdash t : \forall\alpha.\, \tau \quad \Xi \vdash \sigma}{\Xi \mid \Gamma \vdash t\,\sigma : \tau[\alpha := \sigma]}$$

$$\ldots$$

Notes

- $\Xi = \alpha_1, \ldots, \alpha_n$ is a *type context*
- $\Gamma = x_1 : \tau_1, \ldots, x_k : \tau_k$ is a *term context*
- Well-formedness

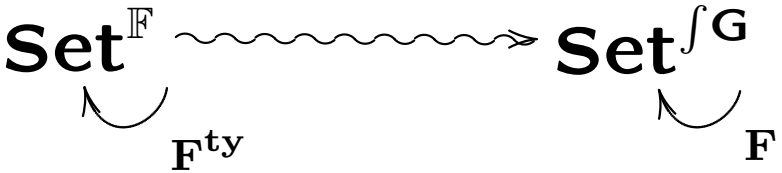
$$\Xi \mid \Gamma \vdash t : \tau \quad \text{is well-formed}$$
$$\Leftrightarrow \Xi \vdash \Gamma \quad \text{and} \quad \Xi \vdash \tau$$

How to Formulate Syntax of $\mathbf{F}$

- Three steps

(2) Contexts

$$\mathbf{G}(n) = \mathbb{F}{\downarrow}(\mathbb{T}(n)) \times \mathbb{T}(n)$$



(1) Types $\mathbb{T}$

(3) $\mathbf{T}$:System $\mathbf{F}$ terms

(1) Polymorphic Types

$$\frac{1 \leq i \leq n}{\alpha_1, \ldots, \alpha_n \vdash \alpha_i} \qquad \frac{}{\alpha_1, \ldots, \alpha_n \vdash b}$$

$$\frac{\alpha_1, \ldots, \alpha_n \vdash \sigma \quad \alpha_1, \ldots, \alpha_n \vdash \tau}{\alpha_1, \ldots, \alpha_n \vdash \sigma \Rightarrow \tau} \qquad \frac{\alpha_1, \ldots, \alpha_n, \alpha_{n+1} \vdash \tau}{\alpha_1, \ldots, \alpha_n \vdash \forall \alpha_{n+1}. \tau}$$

- Aim: Construct types $\mathbb{T} \in \mathbf{Set}^{\mathbb{F}}$
- Endofunctor $\mathbf{F}^{\mathbf{ty}} : \mathbf{Set}^{\mathbb{F}} \longrightarrow \mathbf{Set}^{\mathbb{F}}$ for types by

$$\mathbf{F}^{\mathbf{ty}}(A) = \mathbb{V} + \mathbf{B} + A \times A + \delta A.$$

- Initial $\mathbf{F}^{\mathbf{ty}}$-algebra $(\mathbb{T}, \mathsf{in})$

$$\mathbb{T}(n) = \{\tau \mid \alpha_1, \ldots, \alpha_n \vdash \tau\}$$

(2) Contexts

- Strategy to model terms: define the category, similar to

$$(\mathbf{Set}^{\mathbb{F}\downarrow U})^{U} \simeq \mathbf{Set}^{(\mathbb{F}\downarrow U)\times U}$$

- Choose $U \in \mathbf{Set}$ in terms of $\mathbb{T} \in \mathbf{Set}^{\mathbb{F}}$
- Attempt 1: $U = \coprod_{n\in\mathbb{N}} \mathbb{T}(n)$

$$\mathbf{Set}^{(\mathbb{F}\downarrow \coprod_{n\in\mathbb{N}} \mathbb{T}(n))\times \coprod_{n\in\mathbb{N}} \mathbb{T}(n)}$$

- But the index n does not synchronize
- Should be equal: (modelling $\Xi = n$)

$$\Xi \mid \Gamma \vdash t : \tau \quad \text{is well-formed}$$
$$\Leftrightarrow \Xi \vdash \Gamma \quad \text{and} \quad \Xi \vdash \tau$$

(2) Contexts

- Strategy: choose something similar to

$$(\mathbf{Set}^{\mathbb{F}\downarrow U})^{U} \simeq \mathbf{Set}^{(\mathbb{F}\downarrow U)\times U}$$

- Attempt 2:

$$\mathbf{Set}^{\coprod_{n\in\mathbb{N}}(\mathbb{F}\downarrow\mathbb{T}(n)\times\mathbb{T}(n))}$$

- Insufficient: this does not model renaming between two terms in different type contexts n and n'.

(2) Contexts

- Strategy: choose something similar to

$$(\mathbf{Set}^{\mathbb{F}\downarrow U})^{U} \simeq \mathbf{Set}^{(\mathbb{F}\downarrow U)\times U}$$

- Right way: the Grothendieck construction

$$\mathbf{Set}^{\int \mathbf{G}}$$

- where $\mathbf{G} : \mathbb{F}^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ is a functor

$$\mathbf{G}(n) \stackrel{def}{=} \mathbb{F}\downarrow(\mathbb{T}(n)) \times \mathbb{T}(n)$$

parameterised by a type context n

The Grothendieck Construction

▷ A construction from

an Indexed category to a Category

$$\mathcal{F} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat} \quad \longmapsto \quad \int \mathcal{F} \quad \in \mathbf{Cat}$$

▷ Idea: Construct a single category
to "glue" all indexed categories together, like

$$\begin{aligned} \int \mathcal{F} \quad &\approx \quad \sum_{I \in \mathcal{C}} \mathcal{F}(I) \;=\; \mathcal{F}(I_1) + \mathcal{F}(I_2) + \cdots \\ &= \quad \{(I, A) \mid I \in \mathcal{C},\ A \in \mathcal{F}(I)\} \end{aligned}$$

The Grothendieck Construction

▷ A construction from

an Indexed category to a Category

$$\mathcal{F} : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Cat} \quad \longmapsto \quad \int \mathcal{F} \quad \in \mathbf{Cat}$$

▷ Def. The Grothendieck construction of $\mathcal{F}$ is a category $\int \mathcal{F}$

- objects: (I, A) where $I \in \mathcal{C}$ and $A \in \mathcal{F}(I)$,
- arrows: $(u, \gamma) : (I, A) \longrightarrow (J, B)$ where
  - $u : I \longrightarrow J$ in $\mathcal{C}$ and
  - $\gamma : \mathcal{F}(u)(A) \longrightarrow B$ in $\mathcal{F}(J)$.

Category of Context-with-Types

- $\mathbf{G} : \mathbb{F}^{\mathrm{op}} \longrightarrow \mathbf{Cat}\,; \quad \mathbf{G}(n) \stackrel{def}{=} \mathbb{F}\downarrow(\mathbb{T}(n)) \times \mathbb{T}(n)$
- Category $\int \mathbf{G}$
  - objects $(n \mid \Gamma \vdash \tau)$
  - arrows $(\rho, \pi) : (m \mid \Gamma \vdash \tau) \longrightarrow (n \mid \Delta \vdash \sigma)$
    - renaming $\rho : m \longrightarrow n$ between type contexts
    - renaming $\pi : (\mathbb{F}\downarrow\mathbb{T}\rho)\,(\Gamma) \longrightarrow \Delta$ between term contexts

    where $\mathbb{T}(\rho)(\tau) = \sigma$
- Gives correct renaming between type and term contexts
- Gives correct renaming between term judgments
- Grothendieck is quite right!

(3) Terms of System $\mathbf{F}$

- Endofunctor $\mathbf{F} : \mathbf{Set}^{\int \mathbf{G}} \to \mathbf{Set}^{\int \mathbf{G}}$ for terms

$$
\begin{aligned}
\mathbf{F}(A)(n \mid \Gamma \vdash \tau) = \quad & \mathbf{V}(n \mid \Gamma \vdash \tau) \\
& + \coprod_{\tau_1, \tau_2 \in \mathbb{T}(n)} (\tau \equiv \tau_1 \Rightarrow \tau_2) \times A(n \mid \Gamma, \tau_1 \vdash \tau_2) \\
& + \coprod_{\sigma \in \mathbb{T}(n)} A(n \mid \Gamma \vdash \sigma \Rightarrow \tau) \times A(n \mid \Gamma \vdash \sigma) \\
& + \coprod_{\tau' \in \mathbb{T}(n+1)} (\tau \equiv \forall(\alpha.\tau')) \times A(n+1 \mid \mathsf{wk}(\Gamma) \vdash \tau') \\
& + \coprod_{\substack{\sigma \in \mathbb{T}(n) \\ \tau' \in \mathbb{T}(n+1)}} (\tau \equiv \tau'[\alpha := \sigma]) \times A(n \mid \Gamma \vdash \forall(\alpha.\tau'))
\end{aligned}
$$

- Dependent polynomial functor [Gambino,Hyland,Kock]
- Preserves ω-colimits

(3) Terms of System $\mathbf{F}$

- Endofunctor $\mathbf{F} : \mathbf{Set}^{\int \mathbf{G}} \longrightarrow \mathbf{Set}^{\int \mathbf{G}}$ for terms

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& + \coprod_{\tau_1, \tau_2 \in \mathbb{T}(n)} (\tau \equiv \tau_1 \Rightarrow \tau_2) \times A(n \mid \Gamma, \tau_1 \vdash \tau_2) \\
& + \coprod_{\sigma \in \mathbb{T}(n)} A(n \mid \Gamma \vdash \sigma \Rightarrow \tau) \times A(n \mid \Gamma \vdash \sigma) \\
& + \coprod_{\tau' \in \mathbb{T}(n+1)} (\tau \equiv \forall(\alpha.\tau')) \times A(n+1 \mid \mathsf{wk}(\Gamma) \vdash \tau') \\
& + \coprod_{\substack{\sigma \in \mathbb{T}(n) \\ \tau' \in \mathbb{T}(n+1)}} (\tau \equiv \tau'[\alpha := \sigma]) \times A(n \mid \Gamma \vdash \forall(\alpha.\tau'))
\end{aligned}
$$

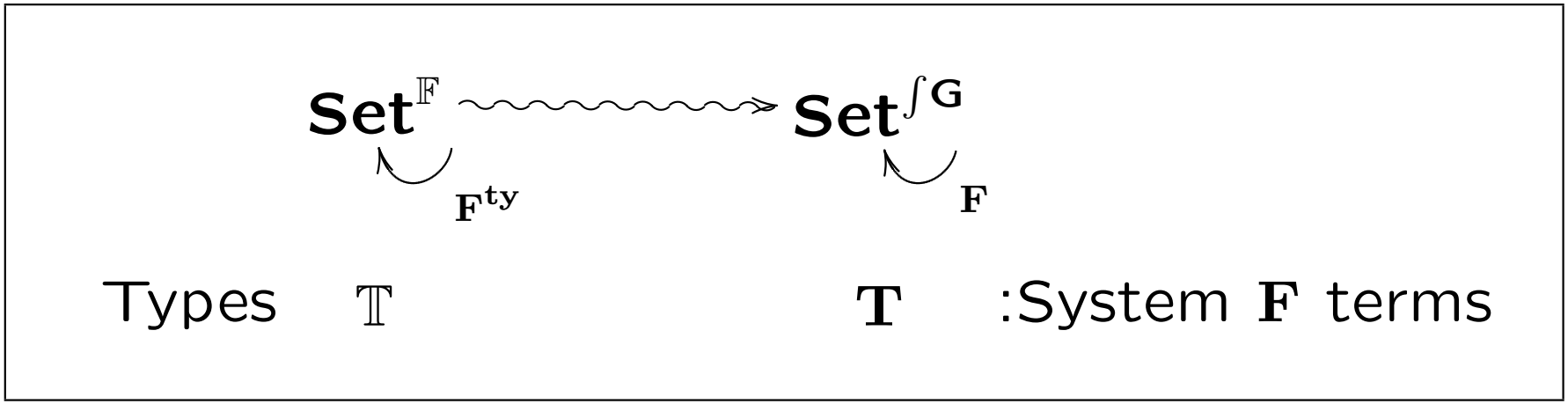
- Theorem

  All system $\mathbf{F}$-terms $\mathbf{T}$ forms an initial $\mathbf{F}$-algebra.

- Can be generic

Summary

- Initial algebra characterisations of polymorphic syntax



  - The Grothendieck construction is a key

Future Work

- Applications
- Dependent type definition in Agda, Coq