Multiversal Polymoprhic Algebraic Theoreis

- Syntax, Semantics, Translations, and Eqational Logic -

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What is Polymorphism

$$\mathrm{id}_A : A \to A$$

$$\mathrm{id} : \forall \alpha . \alpha \to \alpha$$

Polymorphic types = "variable types"

$\alpha \mapsto \mathbb{N}$ $\alpha \mapsto \mathbb{Z}$ $\alpha \mapsto \mathsf{List}(\mathbb{N})$

$$\mathrm{id}_{\mathbb{N}} \qquad \mathrm{id}_{\mathbb{Z}} \qquad \mathrm{id}_{\mathsf{List}(\mathbb{N})}$$

have many instances

Various Polymorphic Sytems

[Girard'71, Reynolds'74]
System F

System Fω

λU
[Girard'72]

ML, Haskell
[Milner'75,78]

System Fc↑
[Yorgey et al.'12]

OO Languages
Java, C++, Scala,etc.

Polymorphic Contracts
[Guha, et al. '07] [Belo et al.'11]

Polymorphic Record Calculus
[Ohori'95]

Polymorphic π-calculus
[Pierce, Sanngiorgi'00]

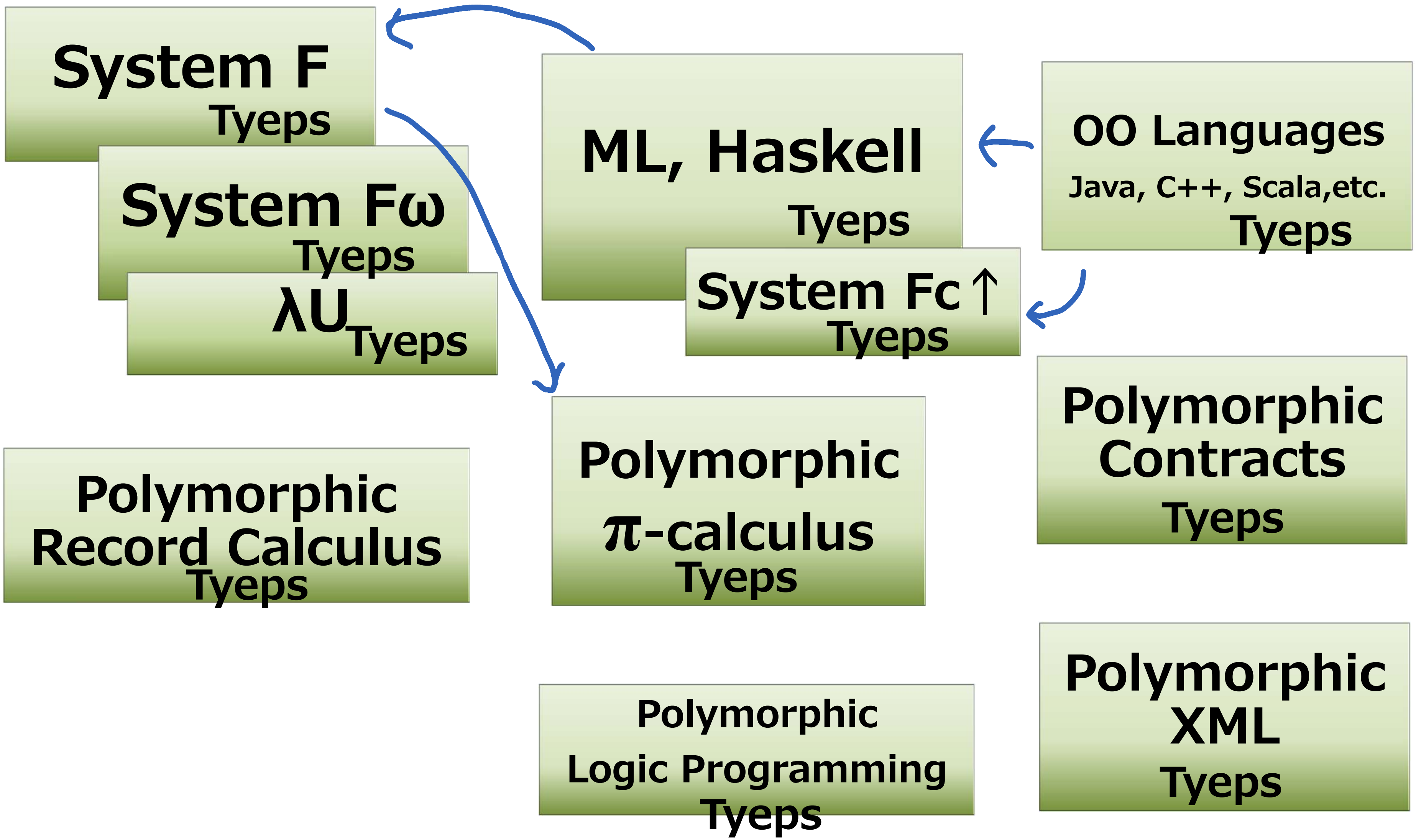
Polymorphic Logic Programming
[Shapiro'91][Hanus'91]

Polymorphic XML
[Hosoya et al.'09]

⇒Unified Theory for Polymorphism

⇒Mutiversal Polymoprhic Algebraic Theoreis

Multiverse of Types



- Universe = collection of all types (cf. Martin-Lof Type Theory)
- Multiverse = multiple type universes
- Translation = algebra homomorphism

Highlights: Categorical Modelling Tools

- Monoids in presheaf categories to model substitutions
  - $\mathbf{Set}^{\mathbb{F}}$ ($\mathbb{F} \cdots$ finite sets & all fun.s)
  - $\mathbf{Set}^{\mathbf{G}U}$
- Polymonial in **Cat**
  Polymonial functor on $\mathbf{Set}^{\mathbf{G}U}$ to model polymorphic terms
- Grothendieck construction
  - to construct polymorphic type contexts

$$\mathbf{G}U = \int^{n \in \mathbb{F}} (\mathbb{F}{\downarrow}Un) \times U(n)$$

  - to capture multiple type universes

$$\mathbf{Poly}(\Sigma) = \int^{U} \mathbf{Poly}(U, \Sigma)$$

How to give Algebraic Theory

- **Signatures, terms, equations**
- Clones in universal algebra
- Lawvere theories
- (Finitary) monads
- Cartesian multicategories

What is an Algebraic Theory

- Universe, types, contexts
- Signatures, terms, equations

Polymorphic Algebraic Theory

Example axioms for System F (Polymorphic λ-calculus)

Vernacular notation

metavariable · object variable · type metavariable · binding · meta-level subst.

$$(\beta) \qquad \Gamma \vdash (\lambda x. M)\, N \;=\; M[x := N] \;:\; \tau$$

$$(\text{type } \beta) \qquad \Gamma \vdash (\Lambda \alpha. M)\, \sigma \;=\; M[\alpha := \sigma] \;:\; \tau[\alpha := \sigma]$$

Formal notation in PEL (Polymoprhic Equational Logic)

$$\mathrm{s}, \mathrm{t} : * \mid \mathrm{M} : (\mathrm{s})\mathrm{t},\ \mathrm{N} : \mathrm{s} \rhd \vdash \mathbf{app}(\mathbf{abs}(x.\, \mathrm{M}[x]), \mathrm{N}) \;=\; \mathrm{M}[\mathrm{N}] \;:\; \mathrm{t}$$

$$\mathrm{s} : *,\ \mathrm{t} : (*)* \mid \mathrm{M} : \langle \alpha \rangle \mathrm{t}[\alpha] \rhd \vdash \mathbf{tapp}\big(\mathbf{tabs}(\alpha.\, \mathrm{M}\lfloor \alpha \rfloor)\big) \;=\; \mathrm{M}\lfloor \mathrm{s} \rfloor \;:\; \mathrm{t}[\mathrm{s}]$$

Theorem

PEL is sound & complete wrt. polymorphic structures.

How to Built Polymorphic Algebirac Theory

Algebraic Characterisation of
Untyped Syntax with Variable Binding
[Fiore, Plotkin, Turi LICS'99]

with Metavariables
[H. APLAS'04]
via Σ-monoids

Polymorphic Abstract Syntax
[H. FoSSaCS'11]
via Grothendieck Constr.

This work

Polymorphic Abstract Syntax with Metavariables

Polymorphic Algebraic Theories

Abstract Syntax and Variable Binding [Fiore,Plotkin,Turi. LICS99]

- Aim: model syntax with variable binding, e.g.

$$\frac{}{x_1,\ldots,x_n \vdash x_i} \qquad \frac{x_1,\ldots,x_n \vdash t \quad x_1,\ldots,x_n \vdash s}{x_1,\ldots,x_n \vdash t@s}$$

$$\frac{x_1,\ldots,x_n,x_{n+1} \vdash t}{x_1,\ldots,x_n \vdash \lambda(x_{n+1}.t)}$$

- Category $\mathbb{F}$ for contexts
  objects: n (contexts) $\{x_1,\ldots,x_n\}$
  arrows: all functions $n \to n'$ (renamings)

$FX = V + X^2 + \delta X$

$\delta X(n) = X(n+1)$

- Category $\mathbf{Set}^{\mathbb{F}} = \mathbb{F} \to \mathbf{Set}$ for modelling terms

$\Lambda \in \mathbf{Set}^{\mathbb{F}}$
$\Lambda(n) = \{\, t \mid n \vdash t \,\}$

functor

$F \circlearrowright \mathbf{Set}^{\mathbb{F}} \quad \Rightarrow \quad \begin{array}{c}\Lambda \\ \| \\ \mu F\end{array} \in \mathbf{Set}^{\mathbb{F}}$

initial F-algebra is abstract syntax with binding

$$\boldsymbol{n} \quad \vdash \quad \boldsymbol{t}$$

context **term**

- $\mathbf{Set}^{\mathbb{F}}$ suffices

Example: System F (Polymorphic λ-calculus)

$$\frac{\overrightarrow{\alpha_i} \mid \overrightarrow{x_j : \tau_j, x : \sigma} \vdash t : \tau}{\overrightarrow{\alpha_i} \mid \overrightarrow{x_j : \tau_j} \vdash \lambda(x : \sigma . t) : \sigma \Rightarrow \tau}$$

$$\frac{\overrightarrow{\alpha_i, \alpha} \mid \overrightarrow{x_i : \tau_i} \vdash t : \tau}{\overrightarrow{\alpha_i} \mid \overrightarrow{x_i : \tau_i} \vdash \Lambda(\alpha . t) : \forall(\alpha . \tau)}$$

Polymorphic Case

$$n \mid \Gamma \vdash t : \tau$$

type context | term context | term | type

Set 

$(n \mid \Gamma \vdash \tau)$

Esp. type univese

$\tau \in U(n)$

⇒Collect all possible dependent triples

Types depend on type variable contexts

$$(n \mid \Gamma \vdash \tau)$$

Def.

A universe of polymorphic types

is

a Σ-monoid $U \in \mathbf{Set}^{\mathbb{F}}$.

signature for type constructors

▷ A **Σ-monoid** [Fiore, Plotkin, Turi LICS'99] for $\Sigma : \mathbf{Set}^{\mathbb{F}} \to \mathbf{Set}^{\mathbb{F}}$ is

- a Σ-algebra $\alpha : \Sigma A \to A$ with
- a monoid in $(\mathbf{Set}^{\mathbb{F}}, \bullet, \mathrm{V})$ $\cdots$ i.e. a cartesian operad

$$\mathrm{V} \xrightarrow{\nu} A \xleftarrow{\mu} A \bullet A$$

where $(A \bullet B)(n) = \int^{m \in \mathbb{F}} A(m) \times B(n)^m, \quad \mathrm{V}(n) = n$

- compatible with the algebra structure.
  - $*$ Unit ν models **object variables**
  - $*$ Multiplication μ models **substitution for object variables**

Category of Discourse for Polymorphic Systems

$$\mathbf{Set}^{\mathsf{G}U}$$

- where U : Σ-monoid

.

$$\mathsf{G}U \stackrel{def}{=} \int^{n \in \mathbb{F}} (\mathbb{F} \downarrow U n) \times U(n)$$

$$(\; n \; | \; \Gamma \; \vdash \; \tau \;)$$

type context

The Grothendieck Construction gives dependent triples.

E.g. $U = \mathcal{M}X \cdots$ syntactic types involving type metavariables X

Polymorphic Algebraic Theory

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Vernacular notation

type metavariable

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meta-level subst.

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$\cap$

free Σ-monoid $\mathcal{M}X$
where $X = \{\mathrm{S}, \mathrm{T}\}$.

Endofunctor for Polymorphic Syntax

signature

Given $\Sigma = (\Sigma^{\mathsf{Ty}}, \Sigma^{\mathsf{Tm}})$

universe = Σ^{Ty}-monoid

U

- Generalised polynomial functor [Fiore ICALP12]

polynomial diagram in Cat

$$P \ : \mathbf{G}U \xleftarrow{s} \ell H \xrightarrow{a} H \xrightarrow{t} \mathbf{G}U$$

[Moerdijk, Palmgren,Gambino,
Hyland,Kock,Batanin,Weber,...]

endofunctor $\Sigma \circlearrowright \mathbf{Set}^{\mathbf{G}U}$

$$\Sigma \stackrel{def}{=} t_! \, a_* \, s^*$$

(U, Σ)-Polymorphic Structures

Def.

General notion of models

Signature $\Sigma = (\Sigma^{\mathsf{Ty}}, \Sigma^{\mathsf{Tm}})$
Universe U
A **(U, Σ)-polymorphic structure** A consists of

(1) endofunctor $\Sigma : \mathbf{Set}^{GU} \to \mathbf{Set}^{GU}$

(2) Σ-algebra $\alpha : \Sigma A \to A$

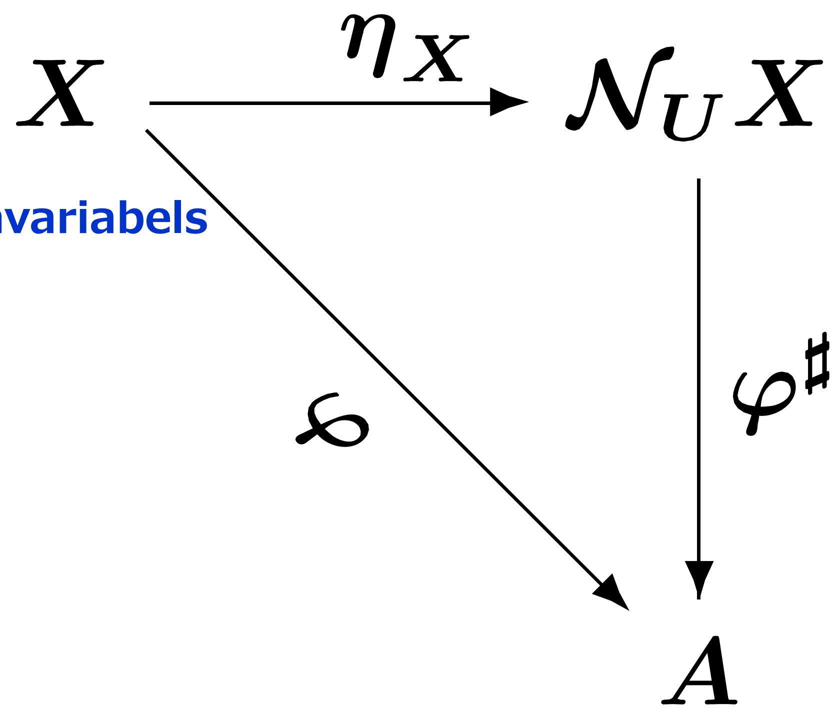
(3) monoid A in $\mathbf{Set}^{GU}$

Σ-monoid

- ▷ unit $\nu : \mathrm{V} \to A$ and
- ▷ multiplication $\mu : A \bullet A \to A$

(4) type-in-term substitution $\varsigma^A : \uparrow A \to A$.

(5) All of these are compatible.



metavariabels

Free (U, Σ)-polymorphic structure
as "syntax with metavariabls"

Homomorphism = interpretation

Category Poly(U, Σ)
- (U, Σ)- polymorphic strucres
- homomorphisms

Any (U, Σ)-polymorphic structure
as "semantics"

- ▶ **We may want to change a universe U in interpretation**

$$\mathrm{Poly}(\Sigma) \stackrel{def}{=} \int^{U} \mathrm{Poly}(U, \Sigma)$$

Summary: Polymorphic Algebraic Theory

Example axioms for System F

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binding

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Various examples
- Existential λ
- Polymorphic FPC
- Algebraic theory for global state

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Paper

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Syntax, Semantics, Translations, and Equational Logic

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`http://www.cs.gunma-u.ac.jp/~hamana/`